

Toric Varieties with NC Toric Actions: NC Type IIA Geometry

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Abstract

Extending the usual \mathbf{C}^{*r} actions of toric manifolds by allowing asymmetries between the various \mathbf{C}^* factors, we build a class of non commutative (NC) toric varieties $\mathcal{V}_{d+1}^{(nc)}$. We construct NC complex d dimension Calabi-Yau manifolds embedded in $\mathcal{V}_{d+1}^{(nc)}$ by using the algebraic geometry method. Realizations of NC \mathbf{C}^{*r} toric group are given in presence and absence of quantum symmetries and for both cases of discrete or continuous spectrums. We also derive the constraint eqs for NC Calabi-Yau backgrounds \mathcal{M}_d^{nc} embedded in \mathcal{V}_{d+1}^{nc} and work out their solutions. The latters depend on the Calabi-Yau condition $\sum_i q_i^a = 0$, q_i^a being the charges of \mathbf{C}^{*r} ; but also on the toric data $\{q_i^a, \nu_i^A; p_I^\alpha, \nu_{iA}^*\}$ of the polygons associated to \mathcal{V}_{d+1} . Moreover, we study fractional D branes at singularities and show that, due to the complete reducibility property of \mathbf{C}^{*r} group representations, there is an infinite number of fractional D branes. We also give the generalized Berenstein and Leigh quiver diagrams for discrete and continuous \mathbf{C}^{*r} representation spectrums. An illustrating example is presented.

Key words: Gauged Linear Sigma Models, Toric Varieties and Calabi-Yau manifolds, Non Commutative Geometry, NC \mathbf{C}^{*r} toric group, NC Toric Varieties and NC Calabi-Yau manifolds, Fractional D -Branes.

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1 Introduction

Matrix model compactification of M theory on non commutative (NC) torii [1] has opened an increasing interest in the study of non commutative spaces, in relation with NC quantum field instantons [2], and open strings of the solitonic sector of type II string theories [3]-[5]. These NC structures have found remarkable applications in various areas of quantum physics such as in the analysis of $D(p-4)/Dp$ brane systems ($p > 3$) [6, 7] and in the study of tachyon condensation using the GMS method [8]. However, most of NC spaces used in these studies involve mainly NC \mathbb{R}_θ^d [9], NC \mathbb{T}_θ^d torii [9, 10], some cases of \mathbb{Z}_n type orbifolds of NC torii [11, 12] and some generalizations to non commutative higher dimensional cycles such as the non commutative extension of Hizerbruch complex surfaces F_n used in [13] and some special Calabi-Yau orbifolds.

Recently efforts have been devoted to go beyond these particular manifolds by considering non commutative extension of complex manifolds with torsion and too particularly NC compact Calabi-Yau manifolds \mathcal{M} because of the basic role they play in type II string compactifications and in the geometric engineering of supersymmetric quiver gauge theories [38]. The most studied examples are given by the class of homogeneous hypersurfaces \mathcal{H}_n embedded in \mathbb{P}^{n+1} projective spaces such as orbifolds of $\mathbb{K}3$ and the quintic \mathcal{Q} [14]-[18], see also [19, 20]. The NC aspect of such varieties is too particularly important for the stringy resolution of singularities offering by the way an alternative method to the standard approach of resolution by deformations of complex and Kahler structures of Calabi-Yau manifolds. The crucial idea in this method is that in NC varieties, the space geometry has a fine structure where the usual commutative zero dimensional points are now represented by (matrix) operators. As a consequence of this deformation commutative space singularities, which are associated with the matrix identity in NC algebra, are naturally lifted; thanks to the spectral partition property of the identity in terms of projectors. Moreover D branes wrapping NC cycles of NC Calabi-Yaus acquire fine structures as well and then fractionate at singularities due to complete reducibility property of matrix identity [21]-[23]. In type II string theory, NC Calabi-Yau manifolds \mathcal{M}^{nc} have moreover a remarkable string states interpretation. The centre of \mathcal{M}^{nc} , which is just the original commutative variety \mathcal{M} , is associated with closed string states while the NC extension is in one to one correspondence with open string states; for details on these aspects and related features see [39].

Recall that the Berenstein and Leigh (BL) idea behind NC Calabi-Yau orbifolds building consists on solving non commutativity in terms of discrete isometries of the orbifolds. This was successfully done in [15]; see also [24], for the study ALE spaces and aspects of the NC quintic; then it has been extended in [17, 18] for the building of NC orbifolds \mathcal{H}_d^{nc} of complex d -dimension homogeneous hypersurfaces \mathcal{H}_d . In the present study, we will push this basic idea a step further by considering a large class of CY manifolds and introducing non commutative toric actions involving NC complex torii. In our construction, we borrow results of BL method but think about points in NC toric manifolds as NC torus fibers based on \mathcal{M} [34]. The NC torii involved here go beyond Connes et al NC "real" torus solution for matrix model compactification and turns out to play a central role in building NC toric manifolds. This way of doing can also be thought of as a first step towards the building of non commutative extension of supersymmetric gauged linear sigma models and their Landau Ginzburg mirrors [40].

To fix the ideas on BL method for NC Calabi-Yau orbifolds with discrete torsion, let us recall one of the useful results on orbifolds of complex d Calabi-Yau homogeneous surfaces \mathcal{H}_d . The latters are described by the homogeneous polynomials $P_d[z_1, \dots, z_{d+2}] = z_1^{d+2} + z_2^{d+2} + z_3^{d+2} + z_4^{d+2} + z_5^{d+2} + a_0 \prod_{\mu=1}^{d+2} z_\mu = 0$ with \mathbf{Z}_{d+2}^d discrete symmetries acting as $z_i \longrightarrow z_i \cdot \omega^{q_i^a}$, where the q_i^a integers satisfy the Calabi-Yau condition $\sum_{i=1}^{d+2} q_i^a = 0$, $a = 1, \dots, d$. Non commutative \mathcal{H}_d^{nc} extending complex d -dimension

hypersurfaces \mathcal{H}_d were shown to be given, in the coordinate patch $Z_{d+2} \sim I_{id}$, by the following non commutative algebra generated by the Z_i operators satisfying,

$$\begin{aligned} Z_i Z_j &= \theta_{ij} Z_j Z_i; & i, j = 1, \dots, (d+1), \\ Z_i Z_{d+2} &= Z_{d+2} Z_i; & i = 1, \dots, (d+1). \end{aligned} \quad (1.1)$$

The θ_{ij} non commutative parameters are solved by discrete torsion as $\theta_{ij} = \omega_{ij} \varpi_{ji}$ with ϖ_{kl} the complex conjugate of ω_{kl} . The ω_{ij} 's are realized in terms of the q_i^a Calabi-Yau charges and the \mathbf{Z}_{d+2}^d discrete group elements $\omega = \exp i \frac{2\pi}{d+2}$ as $\omega_{ij} = \exp i \left(\frac{2\pi}{d+2} m_{ab} q_i^a q_j^b \right) = \omega^{m_{ab} q_i^a q_j^b}$ with m_{ab} integers. But these relations are nothing else than special eqs that describe a special class of complex torii. More general construction are therefore possible.

In this paper, we aim to extend the BL non commutative geometry based on discrete torsion [15],[17],[24],[18], to the large class of complex d dimension Calabi-Yau manifolds \mathcal{M}_d embedded in toric varieties \mathcal{V}_{d+1} [25]-[27]. More precisely, we will focus our attention on NC Calabi-Yau type *IIA* geometries realized as NC hypersurfaces (NC subalgebras) in NC toric manifolds though one might also do similar things for type *IIB* mirrors. This analysis constitutes also the basis for field theoretic construction of NC supersymmetric gauge sigma models and Landau Ginzburg theories. Recall that commutative toric manifolds may, roughly speaking, be thought of as¹ $\mathcal{V}_{d+1} = \mathbf{C}^{k+1} / \mathbf{C}^{*(k+1-d)}$ and are locally parameterized by the coordinates $\{x_i; 0 \leq i \leq k\}$ with $\mathbf{C}^{*(k+1-d)}$ toric actions². This class of \mathcal{M}_d 's is described by hypersurfaces in \mathcal{V}_{d+1} but the underlying polynomials $P_d[x_1, \dots, x_{k+1}]$ defining \mathcal{M}_d are no longer homogeneous contrary to the quintic and \mathcal{H}_d cases. However and even though ignoring discrete torsion, NC extensions of \mathcal{M}_d may be still obtained by endowing the $\mathbf{C}^{*(k+1-d)}$ toric group with a non commutative structure. As we will explicitly show in section 3, the non commutative structure of resulting \mathcal{M}_d^{nc} manifolds is indeed induced by the symmetry of the \mathbf{C}^{*r} toric actions and give the basis of a more general class NC manifolds namely the NC toric varieties. We will show moreover that solutions for non commutative geometry have, in addition to what we were expecting, contributions coming from the toric data $\{q_i^a; \nu_i^A; p_\alpha^I; \nu_{\alpha A}^*\}$ of the polygon $\Delta(\mathcal{M}_d)$ of \mathcal{M}_d . The solutions for the NC constraint eqs we will derive, follow naturally from the toric geometry identities $\sum_i q_i^a = 0$ and $\sum_i q_i^a \nu_i^A = 0$, with the integers q_i^a being the charges of the $\mathbf{C}^{*(k+1-d)}$ toric actions and the integers (ν_i^A) the vertices of the polytope $\Delta(\mathcal{V}_{d+1})$. Furthermore, due to general results on representation theory of the abelian \mathbf{C}^{*r} toric group, we have here also fractional D branes at the singular points of the toric actions; but with the remarkable property that now there are infinitely many. The point is that, like for the case of complex N dimension homogeneous Calabi-Yau orbifolds with \mathbf{Z}_{N+2}^N discrete groups, the NC coordinates of the D branes at the singular points are proportional to the identity operator I_{id} of the toric group representation $\mathcal{R}(\mathbf{C}^{*r})$. As the latters are completely reducible, the identity I_{id} of $\mathcal{R}(\mathbf{C}^{*r})$ is then decomposable into an infinite series in the π_n , ($\pi(\alpha)$ for the continuous case) projectors on the states of the representation space of $\mathcal{R}(\mathbf{C}^{*r})$.

The organization of this paper is as follows; In section 2, we review general aspects of Calabi-Yau manifolds \mathcal{M}_d embedded in toric varieties \mathcal{V}_{d+1} and focus on the study of the type *IIA* geometry of Calabi-Yau manifolds. Similar analysis may be also done for the type *IIB* geometry of \mathcal{M}_d ; the dual of the type *IIA*. Section 3 is mainly devoted to the study of the NC type *IIA* geometry extension of \mathcal{M}_d

¹Toric manifolds are generally defined by cosets $(\mathbf{C}^{k+1} - U) / \mathbf{C}^{*r}$ where $U \subset \mathbf{C}^{k+1}$ is defined by the \mathbf{C}^{*r} actions and a chosen a triangulation. In toric geometry, elements of U are defined by those subsets of vertices, which do not lie together in a single toric cone.

²The \mathbf{C}^{*r} toric group may, roughly speaking, be thought of as a complexification of the $U(1)^r$ gauge symmetry of two dimensional $N = 2$ supersymmetric linear sigma model. This is a continuous abelian group whose representations have an elliptic sector and are gross-modo similar to those of $U(1)^r$ representations.

by help of torsion of the C^{*r} toric actions of the toric varieties. To that purpose, we will first derive the constraint eqs for \mathcal{M}_d^{nc} . Next we analyze different realizations for C^{*r} toric torsions by using quantum symmetries generated by shift operators of the C^{*r} toric group; but also by introducing torsion among the U_a generators of the C^* abelian factors of the C^{*r} group representations. Then we build general solutions of the NC constraint eqs by using effectively both of these two kinds of the C^{*r} torsions. We end this section by giving an illustrating example treating the NC type *IIA* geometry. In section 4, we study fractional branes at the singular points of the toric action and show that, here also, we have fractional D branes at singularities; but with the basic difference that now the identity I_{id} of the representation of each C^* factor of the C^{*r} group is decomposed into an infinite set of projectors on the representation space states. The dimension of fractional D branes are shown to be dependent on the choice of the Calabi-Yau charges of the C^{*r} toric actions of the toric variety \mathcal{V} . In section 5, we give our conclusion and perspectives.

2 Calabi-Yau Hypersurfaces in Toric Varieties

There are different ways for building complex d dimension Calabi-Yau manifolds M . A way to do is by help of 2d $\mathcal{N} = 2$ supersymmetric gauged linear sigma models or again by embedding M in a toric variety \mathcal{V} [36,37]. The latter is a complex Kahler manifold with some C^{*r} toric actions generalizing the usual complex n dimension projective spaces \mathbb{CP}^n . The simplest Calabi-Yau example is given by the case where M is described by complex d dimension hypersurface in a complex $(d+1)$ toric variety \mathcal{V}_{d+1} . To write down algebraic geometry eqs for the Calabi-Yau hypersurfaces; one should specify a number of ingredients namely a local holomorphic coordinates patch of the toric manifold \mathcal{V}_{d+1} , the group of toric action and the toric data. To do so, one should moreover distinguish between two kinds of geometries for the Calabi-Yau manifolds M_d : (1) Type *IIA* geometry, to which we will refer here below to as \mathcal{M}_d , and (2) its type *IIB* mirror geometry often denoted as \mathcal{W}_d . The latter is obtained from \mathcal{M}_d by exchanging their Kahler and complex structures following from the Hodge identities $h^{1,1}(\mathcal{M}_d) = h^{d-1,1}(\mathcal{W}_d)$ and $h^{1,1}(\mathcal{W}_d) = h^{d-1,1}(\mathcal{M}_d)$ [28, 29]. In this study, we will mainly focus our attention on the type *IIA* geometry.

Type *IIA* geometry is constructed in terms of two dimensional $\mathcal{N} = 2$ supersymmetric linear sigma models as follows: First consider a superfield system $\{V_a, X_i\}$ containing r gauge $N = 2$ abelian multiplets $V_a(\sigma, \theta, \bar{\theta})$ with gauge group $U(1)^r$ and $(k+1)$ chiral matter superfields $X_i(\sigma, \theta, \bar{\theta})$ of bosonic components x_i . In addition to the usual terms namely

$$\mathcal{S}[V_a, X_i] = \sum_i \int d^2\theta d^2\bar{\theta} K \left(X_i^* e^{2q_i^a V_a} X_i \right),$$

with K being the gauge covariant Kahler superpotential and q_i^a the charges of X_i under the $U(1)$ s, the linear sigma model action $\mathcal{S}[V_a, X_i]$ of these fields may have r Fayet Iliopoulos (FI) D -terms,

$$\zeta_a \int d^2\sigma d^2\theta d^2\bar{\theta} V_a(\sigma, \theta, \bar{\theta}),$$

with ζ_a being the FI coupling constants. The superfields action $\mathcal{S}[V_a, X_i]$ may also have a holomorphic superpotential $W(X_0, \dots, X_k)$ given by polynomials in the X_i 's, which in the infrared limit, is known to describe a two dimensional conformal field theory describing the string propagation on the type *IIA* background [29]. Let us discuss a little bit this particular geometry.

2.1 Type IIA Geometry

In the method of toric geometry, where to each complex bosonic field x_i it is associated some toric data $\{q_i^a, \nu_i\}$, or more generally by taking into account the data of the dual geometry $\{q_i^a, \nu_i; p_\alpha^I, \nu_\alpha^*\}$ [19, 20]-[25], with ν_i and ν_α^* being $(d+1)$ dimension vectors of \mathbf{Z}^{d+1} self dual lattice, one can write down the algebraic geometry equation of the complex d Calabi-Yau \mathcal{M}_d manifold. This is given by a holomorphic polynomial in the x_i 's with some abelian complex symmetries. In the simplest situation where the toric manifold is given by the coset $\mathcal{V}_{d+1} = \mathbf{C}^{k+1}/C^{*r}$, $d = k - r$, the complex d dimension Calabi-Yau hypersurface reads as,

$$P_d[x_0, \dots, x_k] = b_0 \prod_{i=0}^k x_i + \sum_{\alpha} b_{\alpha} \prod_{i=0}^k x_i^{n_{\alpha i}}. \quad (2.1)$$

where the b_{α} 's are complex structures of \mathcal{M}_d and where the $n_{\alpha i}$ powers are some positive integers constrained by the C^{*r} invariance. Indeed, under the C^{*r} toric action on the \mathbf{C}^{k+1} local coordinates,

$$x_i \longrightarrow x_i \lambda_a^{q_i^a}$$

with q_i^a some integers, the same as in the action $\mathcal{S}[V_a, X_i]$, invariance of $P_d[x_0, \dots, x_k]$ requires the $n_{\alpha i}$ integers are such that,

$$\sum_i q_i^a n_{\alpha i} = 0; \quad \sum_i q_i^a = 0. \quad (2.2)$$

Eqs $\sum_i q_i^a = 0$ follow from the C^{*r} symmetry of the $\prod_{i=0}^k x_i$ monomial while $\sum_i q_i^a n_{\alpha i} = 0$ come from invariance of $\prod_{i=0}^k x_i^{n_{\alpha i}}$ monomials.

2.1.1 Algebraic geometry eqs

Setting $u_{\alpha} = \prod_{i=0}^k x_i^{n_{\alpha i}}$, the above eq (2.1) can be rewritten as $P_d[u_{\alpha}] = \sum_{\alpha} b_{\alpha} u_{\alpha}$, where the u_{α} 's are the effective local coordinates of the coset space $\mathbf{C}^{k+1}/\mathbf{C}^{*r}$. As the u_{α} variables are given by $u_{\alpha} = \prod_{i=0}^k x_i^{n_{\alpha i}}$, it may happen that not all of the u_{α} 's are independent variables; some of these u_{α} , say u_{α_I} for $I = 1, \dots, r^*$, are expressed in terms of the other u_{α_I} variables with $J \neq I$. In other words; one may have relations type $\prod_{\alpha} u_{\alpha}^{p_{\alpha}^I} = 1$, where p_{α}^I are some integers. Substituting $u_{\alpha} = \prod_{i=0}^k x_i^{n_{\alpha i}}$ back into $\prod_{\alpha} u_{\alpha}^{p_{\alpha}^I} = 1$, we discover extra constraint eqs on the $n_{\alpha i}$ and p_{α}^I integers namely,

$$\sum_{\alpha} p_{\alpha}^I n_{\alpha i} = 0. \quad (2.3)$$

In toric geometry the $n_{\alpha i}$ integers are realized as scalar products, $n_{\alpha i} = \langle \nu_i, \nu_{\alpha}^* \rangle = \sum_A \nu_i^A \nu_{\alpha A}^*$, of the toric data vector vertices ν_i and ν_{α}^* of integer entries ν_i^A and $\nu_{\alpha A}^*$ respectively. In this representation, eqs(2.2) and (2.3) are automatically solved by requiring the toric data of the Calabi-Yau manifold to be such that,

$$\sum_i q_i^a \nu_i = 0; \quad \sum_{\alpha} p_{\alpha}^I \nu_{\alpha}^* = 0. \quad (2.4)$$

Let us illustrate these relations for the case of the asymptotically local Euclidean (ALE) space with A_{m-1} singularity. This is a complex two dimension $\mathbf{C}^{n+1}/\mathbf{C}^{*(n-1)}$ toric variety with a $su(m)$ singularity $u_1 u_2 = u_0^m$ at the origin. From this relation, which can be also rewritten as $u_0^{-m} u_1 u_2 = 1$, one sees that there are three effective variables u_0 , u_1 and u_2 ; but only two of them are independent. Since there is one relation, the integer $r^* = 1$ and so there is only one p_{α}^I vector of entries $p_{\alpha} = (-m, 1, 1)$ and three ν_{α}^* vectors given by $\nu_0^* = (1, 0)$, $\nu_1^* = (m, -1)$ and $\nu_2^* = (0, 1)$. More generally, we have the following cases: (a) In the simplest case where no relation such as $\sum_{\alpha} p_{\alpha}^I \nu_{\alpha}^* = 0$ exist, that is all the u_{α} 's are

independent, the algebraic eq of the hypersurface \mathcal{M}_d reads simply as $P_d[u_0, \dots, u_d] = \sum_{\alpha=0}^d b_\alpha u_\alpha = 0$ with $d = (k - r)$. (b) In the generic cases where there are for instance r^* constraint eqs of type $\sum_\alpha p_\alpha^I \nu_\alpha^* = 0$, the $(d + r^* + 1)$ complex variables u_α are not all of them independent and so the algebraic geometry eqs defining the hypersurface embedded in \mathcal{V}_{d+1} is

$$\sum_{\alpha=0}^{d+r^*} b_\alpha u_\alpha = 0; \quad \prod_{\alpha=0}^{d+r^*} u_\alpha^{p_\alpha^I} = 1; \quad I = 1, \dots, r^*, \quad (2.5)$$

where p_α^I are the integers in eqs(2.4).

In the field theoretic language of the two dimensional $N = 2$ supersymmetric linear sigma model with superfields $\{V_a, X_i, 1 \leq a \leq r; 0 \leq i \leq k\}$, the q_i^a integers of the \mathbf{C}^{*r} toric action are the $a - th$ $U(1)$ charge of the x_i fields. Recall in passing that the $U(1)^r$ gauge symmetry group acts on the x_i bosonic fields as $x_i \longrightarrow x_i \exp(iq_i^a \alpha_a)$, with α_a being the gauge group parameters encountered earlier. The condition that the $N = 2$ theory has an extra R -symmetry [29] is effectively given by the Calabi-Yau condition $\sum_{i=0}^k q_i^a = 0$, $a = 1, \dots, r$. Moreover, for the simple case where the $N = 2$ theory has no superpotential ($W(X) = 0$), the moduli space of vacuum configurations minimizing the D -term scalar potential of the $N = 2$ linear sigma model is generated by gauge invariant fields u_α related to the x_i 's as; $u_\alpha = \prod_{i=0}^k x_i^{<\nu_i, \nu_\alpha^*>}$. This gauge invariance or equivalently \mathbf{C}^{*r} toric symmetry, containing as a subgroup the $U(1)^r$ gauge group, of the $u_\alpha = \prod_{i=0}^k x_i^{<\nu_i, \nu_\alpha^*>}$ composite variables follow from $\prod_{i=0}^k \lambda_a^{<q_i^a \nu_i, \nu_\alpha^*>} = 1$ which is exactly solved by the relations $\sum_i q_i^a \nu_i^A = 0$.

The toric manifold $\mathcal{V}_{d+1} = \mathbf{C}^{k+1}/\mathbf{C}^{*r}$ parameterized by the u_α variables is generically singular, but the presence of the FI D-terms $\zeta_a \int d^2\sigma D_a(\sigma)$ into the two dimensional $N = 2$ action $\mathcal{S}[V_a, X_i]$ has the effect of resolving the singularity by blowing up the manifold singularity. To fix the ideas, let us consider the type IIA geometry for the complex dimension 2 ALE space with A_{n-1} singularity. Solving the condition on the dimension, namely $k + 1 - r = 2$ by taking $k = n$ and $r = n - 1$; then using toric geometry method by associating to each moduli x_i the data $\{q_i^a; \nu_i^A; p_\alpha; \nu_{\alpha A}^*\}$ with,

$$\begin{aligned} q_i^1 &= (1, -2, 1, 0, \dots, 0, 0, 0), \quad i = 0, \dots, n, \\ q_i^2 &= (0, 1, -2, 1, \dots, 0, 0, 0), \quad i = 0, \dots, n, \\ &\dots, \\ q_i^{n-1} &= (0, 0, 0, 0, \dots, 1, -2, 1), \quad i = 0, \dots, n, \\ p_\alpha &= (-n, 1, 1) \end{aligned} \quad (2.6)$$

while the vertices are given by,

$$\nu_i = (1, i), \quad \nu_\alpha^* = \begin{pmatrix} 1 & 0 \\ n & -1 \\ 0 & 1 \end{pmatrix}, \quad <\nu_i, \nu_\alpha^*> = (1, n - i, i). \quad (2.7)$$

These data may be expressed in an interesting compact form where one recognizes the structure of the $su(n)$ Cartan matrix as shown here below,

$$\begin{aligned} q_i^a &= \delta_{i-1}^a - 2\delta_i^a + \delta_{i+1}^a, \quad a = 1, \dots, n-1; \quad i = 0, \dots, n, \\ \nu_i &= \nu_i^1 \mathbf{e}_1 + \nu_i^2 \mathbf{e}_2 = \mathbf{e}_1 + i\mathbf{e}_2; \quad i = 0, \dots, n, \\ \nu_\alpha^* &= \nu_{\alpha 1}^* \mathbf{e}_1 + \nu_{\alpha 2}^* \mathbf{e}_2, \quad \alpha = 0, 1, 2, \\ \nu_0^* &= \mathbf{e}_1; \quad \nu_1^* = n\mathbf{e}_1 - \mathbf{e}_2; \quad \nu_2^* = \mathbf{e}_2, \end{aligned} \quad (2.8)$$

where the two vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$ are the generators of the \mathbf{Z}^2 lattice with $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Moreover, the three u_α gauge invariant of the \mathbf{C}^{*n-1} toric action are given by;

$$u_0 = \prod_{i=0}^n x_i; \quad u_1 = \prod_{i=0}^n x_i^{n+1-i}; \quad u_2 = \prod_{i=0}^n x_i^{i-1} \quad (2.9)$$

From this field realization, one sees that these invariants are not all independent since we have the constraint relation

$$u_1 u_2 = u_0^n$$

showing that the complex two dimension toric manifold $\mathbf{C}^{n+1}/\mathbf{C}^{*n-1}$ we are describing has an A_{n-1} singularity at the origin $(u_1, u_2, u_3) = (0, 0, 0)$.

Using this field realization, one may also write down the one dimension hypersurface \mathcal{M}_1 embedded in $\mathbf{C}^{n+1}/\mathbf{C}^{*n-1}$ with A_{n-1} singularity. Its algebraic geometry eq $\sum_{\alpha=0}^2 b_\alpha u_\alpha = 0$ reads in terms of the x_i variables as;

$$P_1 [x_0, \dots, x_n] = b_0 \prod_{i=0}^n x_i + b_1 \prod_{i=0}^n x_i^{n+1-i} + b_2 \prod_{i=0}^n x_i^{i-1} = 0, \quad (2.10)$$

where the b_α 's are complex structure of \mathcal{M}_1 . Invariance of this polynomial under the change $x_i \rightarrow x_i \lambda_a^{q_i^a}$, with $\lambda_a \in \mathbf{C}$, follows from the Calabi-Yau condition $\sum_{i=0}^n q_i^a = 0$, but also from the following obvious identities,

$$\pm \sum_{i=0}^n i q_i^a = \pm (i \delta_{i-1}^a - 2i \delta_i^a + i \delta_{i+1}^a) = 0, \quad (2.11)$$

which are nothing but the relations $\sum_i q_i^a \nu_i^A = 0$.

2.1.2 C^{*r} Toric Symmetry

The λ_a parameters of the C^{*r} toric actions $x_i \rightarrow x_i \lambda_a^{q_i^a}$ are just a kind of complexification of the manifest and familiar $U(1)^r$ gauge symmetry parameters α_a of supersymmetric gauge theories acting on matter as $x_i \rightarrow x_i \exp i \alpha_a q_i^a$. Up to complexifying the $U(1)^r$ symmetry; that is by replacing the α_a real parameters by the complex ones, $\psi_a = \alpha_a - i \rho_a$, $\rho_a \in \mathbf{R}$, the $U(1)^r$ symmetry extends to the \mathbf{C}^{*r} toric actions where now

$$\lambda_a = \exp i \psi_a = \exp (\rho_a + i \alpha_a).$$

As such the $U(1)^r$ gauge symmetry is recovered from the type *IIA* geometry by setting $\rho_a = 0$; i.e $U(1)^r \sim C^{*r}|_{\rho_a=0}$. Therefore the \mathbf{C}^* toric symmetries are given by the cross product of the usual $U(1)$ gauge symmetry with the \mathbf{R}^* group acting as scaling transformations by a real factor $\exp \rho_a$. Contrary to $U(1)$, the \mathbf{R}^* action is not a standard symmetry in unitary field theory; but it plays here a central role and too particularly at the level of moduli space of type *II* string vacuum configurations on Calabi-Yau manifolds. For later use, it is interesting to decompose the \mathbf{C}^* group as,

$$\mathbf{C}^* \sim \mathbf{R}^* \times U(1) \sim U(1) \times \mathbf{R}^*. \quad (2.12)$$

As \mathbf{R}^* and $U(1)$ commute, \mathbf{C}^* representations, $\mathcal{R}(\mathbf{C}^*)$, are mainly given by the tensor product of the \mathbf{R}^* representations $\mathcal{R}(\mathbf{R}^*)$ and the $U(1)$ ones $\mathcal{R}(U(1))$. Moreover like for the $U(1)^r$ symmetry, the \mathbf{C}^{*r} toric group is abelian and the general properties of its representations are grosso-modo similar to those of $U(1)^r$. In practice, the C^{*r} toric group may be defined as given by the set of operators $U_a = \lambda_a^{T_a}$, satisfying

$$U_a U_b = U_b U_a, \quad T_a T_b = T_b T_a, \quad (2.13)$$

and acting on the x_i 's by the following gauge transformations,

$$\begin{aligned} q_i^a x_i &= [T_a, x_i], \\ U_a &: x_i \rightarrow U_a x_i U_a^{-1} = x_i \lambda_a^{q_i^a}. \end{aligned} \quad (2.14)$$

These relations show that r variables x_i among $\{x_1, \dots, x_{k+1}\}$ may be usually fixed to a constant by an appropriate choice of the \mathbf{C}^{*r} gauge parameters. Setting $U_a = \exp(i\psi_a T_a)$; with $iT_a = t_a + i\tau_a$ where t_a and τ_a are hermitian operators and substituting $\psi_a = \alpha_a - i\rho_a$ in the expression of U_a , one gets

$$U_a = \exp i(\rho_a K_a + i\alpha_a Q_a)$$

with $K_a = \frac{\alpha_a}{\rho_a} t_a + \tau_a$ and $Q_a = \tau_a - \frac{\rho_a}{\alpha_a} t_a$ generating \mathbf{R}^{*r} and $U(1)^r$ respectively. Since here $\mathbf{R}^* \times U(1) \sim U(1) \times \mathbf{R}^*$, we have also

$$[K_a, K_b] = 0; \quad [Q_a, Q_b] = 0; \quad [K_a, Q_b] = 0. \quad (2.15)$$

Dilatations generated by K_a and phase transformations generated by Q_b commute; they may be diagonalized simultaneously on a basis of the representation vector space of \mathbf{C}^{*r} .

2.2 More on Type IIA Geometry

From the previous analysis, we have learnt that generic forms of the algebraic equations of the type IIA geometry of complex d -dimension Calabi-Yau manifolds are generally given by the following polynomials,

$$P_d[x_0, \dots, x_k] = b_0 \prod_i x_i + \sum_\alpha b_\alpha \prod_i x_i^{n_{\alpha i}} \quad (2.16)$$

where x_i are holomorphic homogeneous variables satisfying $x_i \rightarrow x_i \lambda_a^{q_i^a}$, the b_α 's are complex structures of the type IIA geometry \mathcal{M}_d of the Calabi-Yau manifold and their number is a priori given by $h^{d-1,1}(\mathcal{M}_d)$. One of the basic property of this algebraic geometry eq is that its invariance under the change $x_i \rightarrow x_i \lambda_a^{q_i^a}$, follows from the Calabi-Yau condition $\sum_i q_i^a = 0$, but also due to the special relations $\sum_i q_i^a n_{\alpha i} = 0$, $a = 1, \dots, r$; $\alpha = 1, \dots$, which have no analogue in the case of isometries of discrete torsions used in [15, 17]. Setting $N_{\alpha i}^a = q_i^a n_{\alpha i}$, this relation may be also rewritten as

$$\sum_{i=0}^k N_{\alpha i}^a = 0, \quad a = 1, \dots, r; \quad \alpha = 1, \dots \quad (2.17)$$

In addition to the $N_{\alpha i}^a$ integers, $N_{\alpha i}^a = q_i^a \nu_i \cdot \nu_\alpha^*$, one also define, out of the toric data $\{q_i^a; \nu_i^A; p_i^I; \nu_\alpha^A\}$, others sets of integers with some specific properties useful when we study the NC toric manifolds. May be the most natural ones are those given by $N_{ij}^{ab} = q_i^a q_j^b \nu_i \cdot \nu_j$ satisfying

$$\sum_{i=0}^k N_{ij}^{ab} = \sum_{j=0}^k N_{ij}^{ab} = 0. \quad (2.18)$$

The N_{ij}^{ab} object is in fact a particular tensor of a more general one namely,

$$N_{ij}^{abAB} = q_i^a q_j^b \nu_i^A \nu_j^B,$$

the latters still obey the previous relations and by summing on the capital indices, one gets N_{ij}^{ab} . Moreover setting $A = B = 1$ and taking into account $\nu_i^1 = 1$, we get the first useful set of integers,

$$L_{ij}^{(1)} = m_{[ab]} q_i^a q_j^b = m_{ab} q_{[i}^a q_{j]}^b, \quad (2.19)$$

where $m_{[ab]} = (m_{ab} - m_{ba})$ is an antisymmetric $r \times r$ matrix with $\frac{r(r-1)}{2}$ entries. The $L_{ij}^{(1)}$ is also a $r \times r$ antisymmetric matrix with $\frac{r(r-1)}{2}$ entries; it satisfies the identities

$$\sum_i L_{ij}^{(1)} = \sum_j L_{ij}^{(1)} = 0$$

inherited from the Calabi-Yau condition $\sum_i q_i^a = 0$. Using the toric data vertices ν_i^A of \mathcal{V} , one may also define an other antisymmetric tensor $L_{ij}^{(2)}$ satisfying as well the identity $\sum_i L_{ij}^{(2)} = \sum_j L_{ij}^{(2)} = 0$ inherited from the condition $\sum_{i=0}^k q_i^a \nu_i^A = 0$. This is given by further contracting the A and B indices of $m_{ab} N_{ij}^{abAB}$ by a tensor metric m_{AB} as shown here below,

$$L_{ij}^{(2)} = Q_i^{aA} Q_j^{bB} (m_{[ab]} m_{(AB)} + m_{(ab)} m_{[AB]}). \quad (2.20)$$

Here m_{ab} is the matrix appearing in eqs(2.19) and m_{AB} is a priori a $(d+1) \times (d+1)$ matrix. Later on, when we study the NC Calabi-Yau manifolds, we will consider only the $m_{(AB)}$ symmetric part and taking it as $m_{AB} = \epsilon_A \epsilon_B$, where ϵ_A are numbers. Therefore $L_{ij}^{(2)}$ have now $(d+1)$ degrees of freedom in addition to those coming from m_{ab} and which were already counted. There is moreover a third term $L_{ij}^{(3)}$ with the same properties as $L_{ij}^{(1)}$ and $L_{ij}^{(2)}$. This term involves quadratic terms in $N_{\alpha i}^a$ and reads as,

$$L_{ij}^{(3)} = N_{\alpha i}^a N_{\beta j}^b (m_{[ab]} m^{(\alpha\beta)} + m_{(ab)} m^{[\alpha\beta]}). \quad (2.21)$$

It satisfies the antisymmetry property $L_{ij}^{(3)} = -L_{ji}^{(3)}$ and the generalized Calabi-Yau condition $\sum_i L_{ij}^{(3)} = 0$ following from $\sum_i N_{\alpha i}^a = 0$. Like for $L_{ij}^{(2)}$, the $m^{\alpha\beta}$ matrix will be taken as $m^{\alpha\beta} = \epsilon^\alpha \epsilon^\beta$. Note that the sum $L_{ij} = L_{ij}^{(1)} + L_{ij}^{(2)} + L_{ij}^{(3)}$ is also an antisymmetric matrix and has the remarkable form $L_{ij} = u_{[i}^a v_{j]}^a$ verifying the the generalized Calabi-Yau identity $\sum_i u_i^a = 0$, $\sum_i v_i^a = 0$ and so;

$$\sum_i L_{ij} = 0. \quad (2.22)$$

In the next section, we will give more details about these special features and the way they enter in the building of the NC type *IIA* geometry. We will mainly focus our attention on Calabi-Yau hypersurfaces \mathcal{M}_d embedded in \mathcal{V}_{d+1} ; but a similar analysis may be also done for the toric variety itself.

3 NC Type *IIA* Geometry

From the algebraic geometry point of view, the NC extension \mathcal{M}_d^{nc} of the Calabi-Yau manifold \mathcal{M}_d , embedded in \mathcal{V}_{d+1} , is covered by a finite set of holomorphic operator coordinate patches $\mathcal{U}_{(\alpha)} = \{Z_i^{(\alpha)}; 1 \leq i \leq k \alpha = 1, 2, \dots\}$ and holomorphic transition functions mapping $\mathcal{U}_{(\alpha)}$ to $\mathcal{U}_{(\beta)}$;

$$\phi_{(\alpha,\beta)} : \mathcal{U}_{(\alpha)} \rightarrow \mathcal{U}_{(\beta)}.$$

This is equivalent to say that \mathcal{M}_d^{nc} is covered by a collection of non commutative local algebras $\mathcal{M}_d^{nc}{}_{(\alpha)}$ generated by the analytic coordinate of the $\mathcal{U}_{(\alpha)}$ patches of \mathcal{M}_d^{nc} , together with analytic maps $\phi_{(\alpha,\beta)}$ on how to glue $\mathcal{M}_d^{nc}{}_{(\alpha)}$ and $\mathcal{M}_d^{nc}{}_{(\beta)}$. The $\mathcal{M}_d^{nc}{}_{(\alpha)}$ algebras have centers $\mathcal{Z}_{(\alpha)} = \mathcal{Z}(\mathcal{M}_d^{nc}{}_{(\alpha)})$; when glued together give precisely the commutative manifold \mathcal{M}_d . In this way a singularity of $\mathcal{M}_d \simeq \mathcal{Z}(\mathcal{M}_d^{nc})$ can be made smooth in the non commutative space \mathcal{M}_d^{nc} [24, 18]. This idea was successfully used to build NC ALE spaces and some realizations of Calabi-Yau threefolds such as the quintic threefolds \mathcal{Q} described by the homogeneous hypersurface [15, 17]

$$P_3[z_1, \dots, z_5] = z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + b_0 \prod_{i=1}^5 z_i.$$

In this context, it was shown that the non commutative quintic \mathcal{Q}^{nc} extending \mathcal{Q} , when expressed in the coordinate patch $Z_5 = I_{id}$, is given by the following special algebra,

$$\begin{aligned}
Z_1 Z_2 &= \alpha Z_2 Z_1, & Z_3 Z_4 &= \beta \gamma Z_4 Z_3, & (a) \\
Z_1 Z_4 &= \beta^{-1} Z_4 Z_1, & Z_2 Z_3 &= \alpha \gamma Z_3 Z_2, & (b) \\
Z_2 Z_4 &= \gamma^{-1} Z_4 Z_2, & Z_1 Z_3 &= \alpha^{-1} \beta Z_3 Z_1, & (c) \\
Z_i Z_5 &= Z_5 Z_i, & i &= 1, 2, 3, 4;
\end{aligned} \tag{3.1}$$

where α, β and γ are fifth roots of the unity, the parameters of the \mathbf{Z}_5^3 discrete group and where the Z_i 's are the generators of \mathcal{Q}^{nc} . One of the main features of this non commutative algebra is that its centre $\mathcal{Z}(\mathcal{Q}^{nc})$ coincides exactly with \mathcal{Q} , the commutative quintic threefolds. In [15], a special solution for this algebra using 5×5 matrices has been obtained and in [17] a class of solutions for eqs(1.1) depending on the Calabi-Yau charges of the quintic threefold has been worked out and partial results regarding higher dimensional Calabi-Yau hypersurfaces were given. A more involved analysis addressing the question of the explicit dependence into the discrete torsion of the orbifold group, the varieties of the fractional D branes at singularities and more generally fractional branes on NC toric manifolds have discussed recently in [18].

One of the key points in the building of \mathcal{Q}^{nc} is the use of discrete torsions of the symmetry $z_i \rightarrow z_i \omega^{q_i^a}$, $\omega^5 = 1$, of the hypersurface $z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + b_0 \prod_{i=1}^5 z_i = 0$. By working in the coordinate patch $z_5 = 1$, then associating to each z_i variable, an operator Z_i with $Z_5 \sim I_{id}$ and finally requiring that the Z_i^5 and $\prod_{i=1}^5 Z_i$ monomials have to be in the centre of \mathcal{Q}^{nc} , one gets constraint eqs when solved give the explicit expression of the Z_i operators in terms of the generators of the orbifold group symmetry of the quintic.

3.1 NC Toric Varieties

To build the non commutative type IIA geometry extending the manifold \mathcal{M}_d , we will more a less adopt the same method introduced in [15, 17]. We start from the complex hypersurface $P_d[x_0, \dots, x_k]$ eq(2.16), with (x_0, \dots, x_k) being the homogeneous variables of $\mathbf{C}^{k+1}/\mathbf{C}^{*r}$. This polynomial has a set of continuous isometries acting on the homogeneous coordinates x_i as $x_i \rightarrow x_i \lambda_a^{q_i^a}$. The main difference between these \mathbf{C}^{*r} actions and the discrete symmetry $z_i \rightarrow z_i \omega^{q_i^a}$ used in building of \mathcal{Q}^{nc} is that its algebraic geometry eq is given by a homogeneous polynomial constraining ω to take a finite set of discrete values, ($\omega^5 = 1$ for \mathcal{Q}). As the polynomial eq(2.16) describing the type IIA geometry is not a homogeneous polynomial, the λ_a 's are arbitrary non zero C -numbers subject to no condition and so one expects emergence of a rich NC structure.

3.1.1 Constraint Eqs

Extending naively the algebraic geometry method used for \mathcal{Q}^{nc} to our present case by associating to each x_i variable the operator Z_i , then taking $q_k^a = 0$ and working in the coordinate patch $x_k = 1$, or equivalently $Z_k = I_{id}$, the NC type IIA geometry \mathcal{M}_d^{nc} may be defined as,

$$\begin{aligned}
Z_i Z_j &= \theta_{ij} Z_j Z_i, & i, j &= 0, \dots, k \\
Z_k Z_i &= Z_k Z_i.
\end{aligned} \tag{3.2}$$

Since \mathcal{M}_d should be in the centre of \mathcal{M}_d^{nc} , it follows that the Z_i generators should satisfy the constraint eqs,

$$\left[Z_i, \prod_{j=0}^k Z_j^{n_{\alpha j}} \right] = 0, \quad (3.3)$$

The non zero θ_{ij} parameters carrying the non commutativity structure of \mathcal{M}_d^{nc} are then constrained as,

$$\prod_{j=0}^k \theta_{ij} = 1, \quad \forall i, \quad \theta_{ij} \theta_{ji} = 1 \quad (3.4)$$

Actually these relations constitute the defining conditions of non commutative type *IIA* geometry \mathcal{M}_d^{nc} . While the constraint relation $\theta_{ij} \theta_{ji} = 1$ shows that $\theta_{ij} = \theta_{ji}^{-1}$, the solution of the constraint eqs $\prod_{j=0}^k \theta_{ij} = 1$ are not trivial and should be expressed in terms of the toric data $\{q_i^a; \nu_i^A; p_\alpha^I; \nu_{\alpha A}^*\}$ of the toric variety.

3.1.2 Comments

We give two comments; the first one concerns the above construction which may be given a deeper explanation. As the Calabi-Yau manifold \mathcal{M}_d is realized as a hypersurface in \mathcal{V}_{d+1} ; it is natural to demand that \mathcal{M}_d^{nc} to be also given by a non commutative subalgebra of a NC toric variety \mathcal{V}_{d+1}^{nc} with \mathbf{C}^{*r} toric actions with *torsions* τ_{ab} . The non commutative structure of \mathcal{V}_{d+1}^{nc} is induced by these torsions and the original commutative toric manifold \mathcal{V}_{d+1} is in its centre; i.e $\mathcal{V}_{d+1} = \mathcal{Z}(\mathcal{V}_{d+1}^{nc})$. Using the previous correspondence $x_i \longrightarrow Z_i$, the NC toric variety may, locally, be defined by the NC algebra

$$Z_i Z_j = \theta_{ij} Z_j Z_i, \quad (3.5)$$

together with

$$\begin{aligned} U_a Z_i &= \mu_{ai} Z_i U_a, \\ U_a U_b &= \vartheta_{ab} U_b U_a, \end{aligned} \quad (3.6)$$

where μ_{ai} and ϑ_{ab} are non zero complex numbers. The NC extension \mathcal{V}_{d+1}^{nc} of the toric variety $\mathbf{C}^{k+1}/\mathbf{C}^{*r}$ may be then thought of as given by $\mathbf{C}_\theta^{k+1}/\mathbf{C}_\tau^{*r}$, the deformation parameters of the NC \mathbf{C}_θ^{k+1} space and the NC \mathbf{C}_τ^{*r} toric group are respectively $\theta = (\theta_{ij})$ and $\tau_{ab} \sim \log \vartheta_{ab}$. The centre $\mathcal{V}_{d+1} = \mathcal{Z}(\mathcal{V}_{d+1}^{nc})$ of the NC toric variety is generated by the \mathbf{C}_τ^{*r} invariants u_α satisfying

$$[Z_i, u_\alpha] = [U_a, u_\alpha] = 0; \quad [u_\alpha, u_\beta] = 0, \quad (3.7)$$

which coincide exactly with eqs(3.3) defining \mathcal{V}_{d+1} .

To summarize \mathcal{M}_d^{nc} is a subalgebra of \mathcal{V}_{d+1}^{nc} and \mathcal{M}_d is contained in the centre $\mathcal{Z}(\mathcal{V}_{d+1}^{nc})$ of the NC toric variety. \mathcal{V}_{d+1}^{nc} and \mathcal{M}_d^{nc} are generated by the Z_i operators while \mathcal{V}_{d+1} and \mathcal{M}_d are generated by the \mathbf{C}_τ^{*r} invariants; the first by the equation $\prod_\alpha u_\alpha^{p_\alpha^I} = 1$, with $u_\alpha = \prod_i Z_i^{n_{\alpha i}}$, and the second by its hypersurface $\sum_\alpha u_\alpha = 0$. Thus we have the following picture,

$$\begin{aligned} \mathcal{M}_d^{nc} &\subset \mathcal{V}_{d+1}^{nc} \\ \mathcal{M}_d &= \mathcal{Z}(\mathcal{M}_d^{nc}) \subset \mathcal{Z}(\mathcal{V}_{d+1}^{nc}). \end{aligned} \quad (3.8)$$

The second comment we want to give deals with extra discrete symmetries of the commutative toric variety \mathcal{V}_{d+1} described by the complex eq $\prod_\alpha u_\alpha^{p_\alpha^I} = 1$. If we denote by ω_α with $\omega_\alpha^{m_\alpha} = 1$, m_α integers

generating a discrete group Γ , and performing the change $u_\alpha \longrightarrow \omega_\alpha^{n_\alpha} u_\alpha$, then invariance of $\prod_\alpha u_\alpha^{p_\alpha^I} = 1$ under Γ requires the following relation to hold,

$$\sum_\alpha n_\alpha p_\alpha^I = 0. \quad (3.9)$$

The simplest example is given by the ALE space with a $su(N)$ singularity described by eq $u_1 u_2 u_0^{-N} = 1$. In this case the discrete group is Z_N , that is $\omega_\alpha = \exp i \frac{2\pi}{N}$ and so the n_α numbers of eq(3.9) are constrained as,

$$n_0 N - n_1 - n_2 = 0 \pmod{N}, \quad (3.10)$$

which is naturally solved by the special choice $n_1 = 1$, $n_2 = -1$ and $n_0 \in \mathbf{Z}$. It is the torsion of this kind of discrete symmetries that has been considered in [24] for building NC ALE spaces. For our concerns, we expect that this relation and above all the relation $\sum_\alpha p_\alpha^I \nu_\alpha^* = 0$ may play an important role in the building of NC type *IIB* geometry using mirror symmetry.

3.2 Solving the Constraint Eqs

First of all note that since the θ_{ij} 's are non zero parameters, one may set

$$\theta_{ij} = \prod_{a,b=1}^r \eta_{ab}^{J_{ij}^{ab}}; \quad \eta_{ab} = \exp(\beta_a \beta_b); \quad \beta_a \in \mathbf{C}, \quad (3.11)$$

and solve the constraint eqs(3.4) by introducing torsions for the \mathbf{C}^{*r} toric actions. Putting this parameterisation back into eqs(3.4), one gets the following constraint on the J_{ij}^{ab} 's,

$$\sum_{i=0}^k J_{ij}^{ab} = 0; \quad J_{ij}^{ab} = -J_{ji}^{ab}. \quad (3.12)$$

Moreover as we are looking for a non commutative structure induced from torsions of the \mathbf{C}^{*r} toric actions and solutions to the Z_i operators as monomials in terms of the \mathbf{C}^{*r} group representation generators, let us start by studying \mathbf{C}^{*r} toric groups with torsions; then turn to build the solutions for eqs(3.4). Lessons from representations of NC torii and orbifolds with discrete torsion teach us that we should distinguish to main cases depending on whether quantum symmetries are taken into account or not.

3.2.1 NC \mathbf{C}^{*r} Toric Actions

The \mathbf{C}^{*r} toric group as used in toric geometry is a complex abelian group which reduces to $U(1)^r$ once the group parameters $\lambda_a = \exp i\psi_a$ are chosen as $|\lambda_a| = 1$, that is $\lambda_a = \exp i\alpha_a$, the α_a 's real numbers and $\rho_a = 0$. The \mathbf{C}^{*r} toric group reduces further to a discrete symmetry $\mathbf{Z}_{N_1} \times \dots \times \mathbf{Z}_{N_r}$ if all α_a 's are chosen as $\alpha_a = \frac{2\pi}{N_a}$, with N_a integers. Therefore, we expect that several features regarding non commutative extensions of the \mathbf{C}^{*r} toric actions to be generalization of known results of NC torii and orbifolds with discrete symmetries. One of the remarkable features concerns the analogue of the quantum symmetry which we want to consider now.

1. Quantum toric symmetry

To better illustrate the introduction of torsion via quantum toric symmetries, we consider the simple case $r = 1$ or again the case of a \mathbf{C}^* factor of the \mathbf{C}^{*r} toric group. Since \mathbf{C}^* is an abelian continuous group and its representations have very special features, we have to distinguish the usual cases; (a) the discrete infinite dimensional spectrum representation and

(b) the continuous one. Both of these realizations are important for the present study and should be thought of as extensions of the irrational representations of NC real torii [10, 12].

a) Discrete Spectrum

Let $\mathcal{R}_{dis}(C^*) = \{U = \exp i\psi T\}$ denote a representation of C^* on an infinite dimensional space \mathbf{E}_{dis} with a discrete spectrum generated by the orthonormal vector basis ³

$$\{|n\rangle; n = (n_1, n_2) \in \mathbf{Z} \times \mathbf{Z} \sim \mathbf{Z}^2\}$$

. Here ψ is a complex parameter, $\psi \in \mathbf{C}$; and T is the complex generator of C^* ; the ψT combination may be split as $\psi T = \rho K + i\alpha Q$, where K is the generator of dilatations and Q is the generator of phases. For $\psi = \alpha$ real ($\rho = 0$), the group representation $\mathcal{R}_{dis}(C^*)$ reduces to $\mathcal{R}_{dis}(U(1)) = \{U = \exp i\alpha Q\}$; the usual $U(1)$ gauge group representation for the $N = 2$ supersymmetric linear sigma model.

The generator T of $\mathcal{R}_{dis}(C^*)$ acts diagonally on the vector basis $\{|n\rangle\}$; i.e. $T|n\rangle = n|n\rangle$ and so the representation group element U acts as $U|n\rangle = (\exp i\psi n)|n\rangle$. The representation elements are also diagonal and read as $U = \sum_n \chi_n(\psi) \pi_n$ where the $\chi_n(\psi)$ characters are given by $\exp(i\psi n)$. The π_n 's, $\pi_n = |n\rangle\langle n|$, are the projectors on the states $|n\rangle$ of the \mathbf{E}_{dis} representation space and may also be expressed as $\pi_n = \sum_n \exp(-i\psi n) U_n$. Since the representation $\mathcal{R}_{dis}(C^*)$ is completely reducible, its I_{id} identity operator may be decomposed in a series of π_n as,

$$I_{id} = \sum_n \pi_n. \quad (3.13)$$

Such a relation should be compared with analogous ones for $U(1)$ and more particularly abelian discrete groups such as the \mathbf{Z}_N symmetries appearing in the well known $\mathbf{C}^2/\mathbf{Z}_N$ orbifolds.

Like for $U(1)$ and the \mathbf{Z}_N discrete symmetries, we have here also a complex shift operator $V_{\tau(1,1)} \equiv V_\tau$ of the C^* group acting on $\{|n\rangle; n \in \mathbf{Z} + i\mathbf{Z}\}$ as an automorphism exchanging the \mathbf{C}^* characters $\chi_n(\psi)$. This translation operator which operates as $V_\tau|n\rangle = |n + \tau\rangle$; with $\tau_{(1,1)} = 1 + i$, may also be defined by help of the $\mathbf{a}_{(n_1, n_2)}^+ = |(n_1 + 1) + in_2\rangle\langle n_1 + in_2|$ and $\mathbf{b}_{(n_1, n_2)}^+ = |n_1 + i(n_2 + 1)\rangle\langle n_1 + in_2|$ step operators as,

$$\begin{aligned} V_{(1,1)} &= V_1 V_i \\ V_1 &= \sum_{n_1, n_2 \in \mathbf{Z}} \mathbf{a}_{(n_1, n_2)}^+; \quad V_i = \sum_{n_1, n_2 \in \mathbf{Z}} \mathbf{b}_{(n_1, n_2)}^+. \end{aligned} \quad (3.14)$$

Due to the remarkable property $\mathbf{a}_{(n_1, n_2)}^+ \pi_{(n_1, n_2)} = \pi_{(n_1+1, n_2)} \mathbf{a}_{(n_1, n_2)}^+$, $\mathbf{b}_{(n_1, n_2)}^+ \pi_{(n_1, n_2)} = \pi_{(n_1, n_2+1)} \mathbf{b}_{(n_1, n_2)}^+$ and $\mathbf{b}_{(n_1, n_2)}^+ \mathbf{a}_{(n_1, n_2)}^+ \pi_{(n_1, n_2)} = \pi_{(n_1+1, n_2+1)} \mathbf{b}_{(n_1, n_2)}^+ \mathbf{a}_{(n_1, n_2)}^+$, it follows that the operators U and V satisfy the following non commutative algebra,

$$UV = e^{-i\psi\tau} VU, \quad (3.15)$$

describing a complex version of the CDS torus [1]; to which we shall refer to as the non commutative complex two torus. Since ψ is an arbitrary complex parameter, eqs(3.15) define an irrational discrete representation of the NC complex two torus. This representation satisfy the following natural relation, useful when we discuss the solution of the constraint eqs(3.4).

$$U^k V^l = (e^{-i\psi\tau})^{kl} V^l U^k. \quad (3.16)$$

Before going ahead, let us make three remarks regarding the representations of the \mathbf{C}^* toric group. The first remark is that as far as the factor $C^* \sim \mathbf{R}^* \times U(1)$ is concerned, one should distinguish four sectors for $\mathcal{R}(C^*)$ according to the spectrums of its subgroup representations $\mathcal{R}(\mathbf{R}^*)$ and $\mathcal{R}(U(1))$:

³We will use the convention notation $n \equiv n_1 + in_2 \in \mathbf{Z} + i\mathbf{Z} \sim (n_1, n_2) \in \mathbf{Z}^2$; $|n\rangle$ should be then thought of as $|n_1\rangle \otimes |n_2\rangle$.

(i) *Discrete-discrete sector* denoted as $\mathcal{R}_{(dis,dis)}(C^*)$, this sector has discrete spectrums for both of the two subgroup representations $\mathcal{R}_{dis}(\mathbf{R}^*)$ and $\mathcal{R}_{dis}(U(1))$ of $\mathcal{R}_{(dis,dis)}(C^*)$; that is $\mathcal{R}_{dis}(\mathbf{R}^*) \sim \mathbf{Z}$ and $\mathcal{R}_{dis}(U(1)) \sim \mathbf{Z}$ and so,

$$\mathcal{R}_{(dis,dis)}(C^*) \sim \mathbf{Z} \times \mathbf{Z} \quad (3.17)$$

This is the case we have discussed above.

(ii) *Discrete-continuous sector*: $\mathcal{R}_{(dis,con)}(C^*)$. Here the two abelian subgroup representations have one discrete and one continuous spectrums either; $\mathcal{R}_{dis}(\mathbf{R}^*) \sim \mathbf{Z}$, and $\mathcal{R}_{con}(U(1)) \sim \mathbf{R}$ and so,

$$\mathcal{R}_{(dis,con)}(C^*) \sim \mathbf{Z} \times \mathbf{R}. \quad (3.18)$$

or a *continuous-discrete sector*: $\mathcal{R}_{(con,dis)}(C^*)$ where now $\mathcal{R}_{con}(\mathbf{R}^*) \sim \mathbf{R}$, but $\mathcal{R}_{dis}(U(1)) \sim \mathbf{Z}$; $\mathcal{R}_{(con,dis)}(C^*) \sim \mathbf{R} \times \mathbf{Z}$. (iii) Finally the *continuous-continuous sector* $\mathcal{R}_{(con,con)}(C^*) \sim \mathbf{R} \times \mathbf{R}$ where both subgroups representations have continuous spectrums. We will give more details on this fourth kind of representations once we finish the remarks.

The second remark concerns the case of C^{*r} toric symmetries generated by the U_a operators and the $V_{\tau_a} = V_a$ automorphisms, eq(3.15) extends, in the simplest situation, as $U_a V_b = \delta_{ab} \exp(-i\psi_a \tau_a) V_b U_a$. More general extensions of eq(3.15) may be also written down; for more details see [18] for similar realizations concerning discrete symmetries. The last remark we give deals with the shift operator V_τ ; like for the C^* group element U , the operator V_τ is an element of the C^* dual group, denoted as \tilde{C}^* , and acting on C^* as $VUV^{-1} = e^{i\psi\tau}U$. So the groups C^* and \tilde{C}^* do not commute; i.e

$$C^* \tilde{C}^* \neq \tilde{C}^* C^* \quad (3.19)$$

Later on we will consider the other case where two different C^* factors of the toric group C^{*r} do not commute as well. For the moment we turn to complete our discussion by describing briefly the continuous spectrums.

b) Continuous Case

In this case the generator T of $\mathcal{R}_{(con,con)}(C^*)$ has a continuous spectrum with a vector basis state $\{|z\rangle, z \in \mathbf{C}; \langle z'|z\rangle = \delta(z - z')\}$ and acts diagonally as $\langle z|T|z\rangle = z$. The representation of the group element U is $\langle z|U = (\exp i\psi z) \langle z|$ which may also be put in the form $U = \int dz \chi(\psi, z) \pi(z)$ where the continuous $\chi(\psi, z)$ character function is given by $\exp(i\psi z)$ and where the $\pi(z)$'s, $\pi(z) = |z\rangle\langle z|$; $\pi(z)\pi(z') = \delta(z - z') \pi(z)$, are the projectors on the $|z\rangle$ states. Since the representation $\mathcal{R}_{(con,con)}(C^*)$ is completely reducible, the I_{id} identity operator may be decomposed as,

$$I_{id} = \int dz \pi(z). \quad (3.20)$$

The shift operator by an ϵ amount, denoted as $V(\epsilon)$, acts on $\{|z\rangle, z \in \mathbf{C}\}$ as $\langle z|V(\epsilon) = \langle z + \epsilon|$. It may be defined, by help of $\mathbf{a}^+(z, \epsilon) = |z\rangle\langle z + \epsilon|$ operators, as,

$$V(\epsilon) = \int dz \mathbf{a}^+(z, \epsilon). \quad (3.21)$$

These operators satisfy similar relations as in eqs(3.8-9) namely $UV = \exp(-i\psi\tau) VU$ and $U^k V^l = (\exp[-i\psi\tau])^{kl} V^l U^k$. This realization may also be defined on the space of holomorphic functions $F(z) = \langle z|F\rangle$ where

$$UFU^{-1} = (\exp i\psi z) F, \quad V_\epsilon F V_\epsilon^{-1} = (\exp \epsilon \partial_z) F \quad (3.22)$$

2. NC C^{*r} toric cycles Here we forget about the \tilde{C}^* quantum symmetry and its V_a generators and focus our attention on the C^* toric generators U_a only. Of course a more general representation should include also the V_a shift operators; it will be given later on; but as far the U_a 's are concerned, one may also build representations where the r complex cycles of the C^{*r} group are non commuting. This is achieved by introducing torsion among the C^* subgroups of the toric symmetry by demanding the T_a generators to not commute; $[T_a, T_b] \neq 0$; say $[T_a, T_b] = im_{[ab]}$. This means that given two toric actions C^* and $C^{*'}$, we have

$$C^* C^{*'} \neq C^{*'} C^* \quad (3.23)$$

Here also we should distinguish between discrete and continuous spectrums. In the particular case of a continuous spectrum, the $U_a = \lambda_a^{T_a} = \exp i\psi_a T_a$, the algebra of NC C^{*r} toric cycles is defined as

$$U_a U_b = \Lambda_{ab}^{m_{[ab]}} U_b U_a; \quad \Lambda_{ab} = \exp(-i\psi_a \psi_b), \quad [T_a, T_b] = im_{[ab]} I_{id}. \quad (3.24)$$

The $r \times r$ matrix m_{ab} carries the C^{*r} group torsion. A possible realization of the generators T_a on the space \mathcal{F} of holomorphic functions $f(y_1, \dots, y_r)$ with r arguments, is that given by,

$$[T_a, f(y_1, \dots, y_r)] = (\partial_a - im_{ac} y_c) f. \quad (3.25)$$

From this realization, one can check that the T_a generators satisfy indeed the algebra $[T_a, T_b] = im_{[ab]} I_{id}$. Note that eigen functions $f(y_1, \dots, y_r)$ of T_a 's with eigenvalues k_a are given by the holomorphic exponentiels $f_{(k_1, \dots, k_r)} = \exp i \frac{m_{ab} k_b y_a^2}{2}$; i.e

$$[T_a, f] = k_a f. \quad (3.26)$$

These functions transform under finite transformations of the C^{*r} symmetry as

$$U_a f_{(k_1, \dots, k_r)} U_a^{-1} = (\exp i\psi_a k_a) f_{(k_1, \dots, k_r)}$$

Other representations of the C^{*r} toric group with torsion may be also written down; they are mainly obtained by complex extensions of known results on non commutative real torii. However and as far as realizations of NC toric group with torsions are concerned, one may introduce also the quantum symmetries by allowing the $f(y_1, \dots, y_r)$ functions to depend on extra variables z_a so that the new function is $F(z_a; y_a)$ and the C^{*r} realization reads as,

$$U_a F U_a^{-1} = (\exp i\psi_a (z_a + \partial_{y_a} - im_{ac} y_c)) F, \quad V_b F V_b^{-1} = (\exp \epsilon_{bd} \partial_{z_d}) F \quad (3.27)$$

Having studied the main lines of NC C^{*r} toric actions, we turn now to solve the constraint eqs (3.4).

3.2.2 Representations of the Z_i 's

The constraint eqs(3.4) may be solved in different ways depending on whether quantum symmetries of the C^{*r} actions are taken into account or not. For the special example where only the U_a generators of the C^{*r} toric group are considered; then we can solve the θ_{ij} parameters by using the m_{ab} torsions. In the general case, one should also be aware about the τ_{ab} torsions between the U_a and V_a generators; i.e by using the relations $U_a V_b = \Omega_{ab}^{\tau_{ab}} V_b U_a$. Here below, we shall give details for the case where $\tau_{ab} = 0$; but $m_{[ab]} \neq 0$.

Representation I

In this representation, the Z_i 's are realized in terms of the $U_a = \exp(i\psi_a T_a)$ as

$$Z_i = x_i \prod_{a=1}^r U_a^{q_i^a} = x_i \prod_{a=1}^r \exp(iq_i^a \psi_a T_a), \quad (3.28)$$

where x_i are complex moduli, which we shall interpret as just the commutative coordinates of the toric manifold \mathcal{V}_{d+1} containing \mathcal{M}_d . Since the NC C^{*r} toric group generators fulfill relations such $U_a^s U_b = \Lambda^{sm_{[ab]}} U_b U_a^s$ and $U_a U_b^t = \Lambda^{tm_{[ab]}} U_b U_a^t$, it follows therefore that $U_a^{q_i^a} U_b^{q_j^b} = \Lambda^{m_{[ab]} q_i^a q_j^b} U_b^{q_j^b} U_a^{q_i^a}$ and consequently

$$\theta_{ij} = \prod_{a,b=1}^r \Lambda_{ab}^{m_{[ab]} q_i^a q_j^b}. \quad (3.29)$$

Putting this solution back into eqs(3.4), one discovers that the constraint eqs are indeed fulfilled because of the Calabi-Yau condition $\sum_{i=0}^k q_i^a = 0$, but also due to the toric data encoded in $\sum_{i=0}^k q_i^a n_{\alpha i} = 0$, as shown here below ,

$$\begin{aligned} \prod_{i=0}^k Z_i &= \left(\prod_{i=0}^k x_i \right) \prod_{a=1}^r U_a^{\sum_{i=0}^k q_i^a} = \left(\prod_{i=0}^k x_i \right) I_{id}, \\ \prod_{i=0}^k Z_i^{n_{\alpha i}} &= \left(\prod_{i=0}^k x_i^{n_{\alpha i}} \right) \prod_{a=1}^r U_a^{\sum_{i=0}^k q_i^a n_{\alpha i}} = \left(\prod_{i=0}^k x_i \right) I_{id}. \end{aligned} \quad (3.30)$$

Before going ahead note that such solutions may be extended by switching on the τ_{ab} torsions. The result is $Z_i = x_i \prod_{a=1}^r (U_a V_a)^{q_i^a}$. Note moreover that according to the spectrums of the group representation of the $U_a = \exp(i\psi_a T_a)$ elements, the Z_i operators may have also discrete or continuous spectrums.

Other Representations

The solutions we have given here above are still less general; they are based on the two following remarkable identities; $\sum_{i=0}^k q_i^a = 0$ and $\sum_{i=0}^k q_i^a n_{\alpha i} = 0$. There are however other relations similar to the previous ones and which can do the same job. These relations are given by the identities $\sum_{i=0}^k Q_i^{aA} = \sum_{i=0}^k N_{i\alpha}^a = 0$ defining the toric data eqs (2.17-18) of the polygon of the Calabi-Yau manifold. Taking into account of these identities, one may write down more general solutions extending eqs(3.28);

$$Z_i = x_i \prod_{a=1}^r \exp i\psi_a \left(q_i^a + \sum_{A=1}^d \epsilon_A Q_i^{aA} + \sum_{\alpha} \epsilon^{\alpha} N_{i\alpha}^a \right) T_a, \quad (3.31)$$

where ϵ_A and ϵ^{α} are in general complex numbers. The last two terms on the right hand of the above equation namely, $\left(\sum_{A=1}^d \epsilon_A Q_i^{aA} + \sum_{\alpha} \epsilon^{\alpha} N_{i\alpha}^a \right) T_a$, constitute extra contributions for the NC commutativity θ_{ij} parameters of the NC Calabi-Yau manifold. They reflect the couplings between the C^{*r} toric action and the toric data of the polygons Δ of the toric manifold.

As a summary to this presentation, the Z_i operator coordinates of the NC Calabi-Yau manifold \mathcal{M}_d^{nc} are generally realized, in terms of the U_a generators of the C^{*r} toric group and their underlying quantum symmetries generated by the V_a shift operators, as follows

$$Z_i = x_i \prod_{a=1}^r (U_a V_a)^{\tilde{q}_i^a}. \quad (3.32)$$

Here \tilde{q}_i^a , which are given by $\tilde{q}_i^a = q_i^a + \sum_{A=1}^d \epsilon_A Q_i^{aA} + \sum_{\alpha} \epsilon^{\alpha} N_{i\alpha}^a$, are a kind of shifted Calabi-Yau charges which satisfy the condition $\sum_{i=0}^k \tilde{q}_i^a = 0$ and their integrality follow by requiring ϵ_A and ϵ^{α} to be integer numbers. Using the relations

$$U_a U_b = \Lambda_{ab}^{m_{[ab]}} U_b U_a$$

and

$$U_a V_b = \Omega_{ab}^{\tau_{ab}} V_b U_a,$$

the non commutative θ_{ij} parameters are expressed as

$$\theta_{ij} = \prod_{a,b=1}^r \Lambda_{ab}^{J_{ij}^{ab}} \Omega_{ab}^{K_{ij}^{ab}},$$

where now $J_{ij}^{ab} = m_{[ab]} \tilde{q}_i^a \tilde{q}_j^b$ and $K_{ij}^{ab} \sim \tau_{[ab]} \tilde{q}_i^a \tilde{q}_j^b$. By appropriate choices of Λ_{ab} , Ω_{ab} , $m_{[ab]}$ and $\tau_{[ab]}$, one recovers as special cases the representations involving discrete torsions obtained in refs [15, 17].

3.3 Example

To illustrate the previous analysis, we consider the NC extension of a Calabi-Yau manifold with a conic singularity. This manifold is defined as a hypersurface \mathcal{M}_2 embedded in the complex three dimension toric variety $\mathcal{V}_3 \subset \mathbf{C}^6 / \mathbf{C}^{*2}$ with a \mathbf{C}^{*2} toric actions defined as $x_i \longrightarrow x_i \exp i(\psi_a q_i^a)$ and q_i^a charges taken as; $q_i^1 = (1, -1, 1, -1, 0, 0)$, $q_i^2 = (1, 0, -1, 1, -1, 0)$ and $p_\alpha = (1, -1, 1, -1)$. The ν_i and ν_α^* vertices of the polygon Δ of the toric manifold and it dual ∇ , satisfying $\sum_{i=1}^6 q_i^a \nu_i = 0$ and $\sum_{\alpha=1}^4 p_\alpha \nu_\alpha^* = 0$, are given the following four dimensional vectors with integer entries $\nu_i^A = (\nu_i^1, \nu_i^2, \nu_i^3, \nu_i^4)$ and $\nu_{iA}^* = (\nu_{i1}^*, \nu_{i2}^*, \nu_{i3}^*, \nu_{i4}^*)$ respectively,

$$\nu_i = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & n_2 + n & n_2 & m \end{pmatrix}, \quad \nu_\alpha^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 2 & -2 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}. \quad (3.33)$$

The four u_α gauge invariants read as $u_\alpha = \prod_{i=1}^6 x_i^{n_{\alpha i}}$, $i = 1, \dots, 6$ with $n_{\alpha i}$ integers given by,

$$n_{\alpha i} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 0 & 2 & n+1 \\ 3 & 3 & 0 & 0 & 3 & 2n+m+1 \\ 2 & 2 & 1 & 1 & 2 & n+m+1 \end{pmatrix}. \quad (3.34)$$

They satisfy the constraint eq $n_{0i} + n_{2i} = n_{1i} + n_{3i}$, showing in turns that the complex three dimension toric manifold \mathcal{V}_3 is described by $u_0 u_2 = u_1 u_3$ having a conic singularity at the origin. The complex two dimension Calabi-Yau hypersurface embedded in \mathcal{V}_3 , $\sum_\alpha u_\alpha = 0$, reads, in terms of the x_i local coordinates, as

$$P(x_1, \dots, x_6) = a \prod_{i=1}^6 x_i + x_1^2 x_2^2 x_6^{n+1} (x_5^2 + b x_1 x_2 x_5^3 x_6^{n+m} + c x_3 x_4 x_5^2 x_6^m), \quad (3.35)$$

where n and m are positive integers which may be fixed to some values.

The non commutative extension \mathcal{M}_2^{nc} of this holomorphic hypersurface is given by the NC algebra (3.2), generated by the Z_i generators satisfying eqs(3.28-31) with $\theta_{ij} = \Lambda^{L_{ij}}$ parameters given by eqs(29). For the special case where the L_{ij} antisymmetric matrix is restricted to $L_{ij} = m(q_i^1 q_j^2 - q_j^1 q_i^2)$, with $L_{ij} = -L_{ji}$ and $L_{i6} = 0$ and entries,

$$L_{ij} = m \begin{pmatrix} 0 & 1 & -2 & 2 & -1 \\ -1 & 0 & 1 & -1 & 1 \\ 2 & -1 & 0 & 0 & -1 \\ -2 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix}. \quad (3.36)$$

The NC complex surface \mathcal{M}_2^{nc} is then given by a one parameter algebra generated by following relations,

$$\begin{aligned}
Z_1 Z_2 &= \Lambda^m Z_2 Z_1, & Z_1 Z_3 &= \Lambda^{-2m} Z_3 Z_1, & Z_1 Z_4 &= \Lambda^{2m} Z_4 Z_1, \\
Z_1 Z_5 &= \Lambda^{-m} Z_5 Z_1, & Z_2 Z_3 &= \Lambda^m Z_3 Z_2, & Z_2 Z_4 &= \Lambda^{-m} Z_4 Z_2, \\
Z_2 Z_5 &= \Lambda^m Z_5 Z_2, & Z_3 Z_4 &= Z_4 Z_3, & Z_3 Z_5 &= \Lambda^{-m} Z_5 Z_3, \\
Z_4 Z_5 &= \Lambda^m Z_5 Z_4, & Z_i Z_6 &= Z_6 Z_i,
\end{aligned} \tag{3.37}$$

where Λ^m is given by $\Lambda^m = \exp(-im\psi_1\psi_2)$. Since $\psi_a = \rho_a - i\alpha_a$; it follows that $m\psi_1\psi_2 = m(\rho_1\rho_2 - \alpha_1\alpha_2) - im(\alpha_1\rho_2 + \alpha_2\rho_1)$ which we set as $\Lambda^m = \exp(\kappa + i\phi)$ for simplicity. This is a complex parameter enclosing various special situations corresponding to: (1) *Hyperbolic representation* described by $(\kappa, \phi) \equiv (\kappa, 0)$; it corresponds to torsions induced by $\mathbf{R}^{*2} \otimes U(1)^2$ subgroup of the C^{*2} toric group. (2) *Periodic representations* corresponding to $(\kappa, \phi) \equiv (0, \phi + 2\pi)$ where $|\Lambda^m| = 1$. It is associated with a NC $\mathbf{R}^* * U(1)$ toric actions of the C^{*2} group. (3) *Discrete periodic representations* $(\kappa, \phi) \equiv (0, N\phi + 2\pi)$ with $|\Lambda^m| = 1$ but moreover $(\Lambda^m)^N = 1$. This last case is naturally a subspace of the periodic representation and it is precisely the kind of situation that happen in the building of NC manifolds with discrete torsion. It is associated with the \mathbf{Z}_N subgroup of $U(1)$.

4 Fractional Branes

The NC type *IIA* realization we have studied so far concerns regular points of the algebra, that is non singular ones of the \mathbf{C}^{*r} toric group. In this section, we want to complete this analysis by considering the representations at C^{*r} group fixed points where it is expected to get fractional D branes. To do so, we shall first classify the various \mathcal{M}_d subsets $\mathcal{S}_{(\mu)}$ of stable points under \mathbf{C}^{*r} ; then we give the quiver diagrams extending those of Berenstein and Leigh obtained for the case of orbifolds with discrete torsion. The method is quite similar to that of [24].

4.1 Fixed points of the C^{*r} toric actions

To fix the idea, consider that class of Calabi-Yau manifolds \mathcal{M}_d described by hypersurfaces $P_d[x_1, \dots, x_{k+1}]$ eq(2.16) embedded in \mathcal{V}_{d+1} , with toric data $\{q_i^a, \nu_i; p_\alpha^I, \nu_\alpha^*\}$ satisfying eqs(2.4). The local holomorphic coordinates $\{x_i \in \mathbf{C}^{k+1}; \quad 0 \leq i \leq k\}$ are not all of them independent as they are related by the \mathbf{C}^{*r} gauge transformations $U_a : x_i \longrightarrow U_a x_i U_a^{-1} = x_i \lambda_i^{q_i^a}$, with $\sum_{i=0}^k q_i^a = 0$. Fixed points of the \mathbf{C}^{*r} gauge transformations are given by the solutions of the constraint eq

$$x_i = U_a x_i U_a^{-1} = x_i \lambda_i^{q_i^a}. \tag{4.1}$$

From this relation, one sees that its solutions depend on the values of q_i^a ; the x_i 's should be zero unless $q_i^a = 0$. Fixed points of \mathbf{C}^{*r} toric actions are then given by the \mathcal{S} subspace of \mathbf{C}^{k+1} whose x_i local coordinates are \mathbf{C}^{*r} gauge invariants. To get a more insight into this subspace it is interesting to note that as $\mathbf{C}^{k+1}/\mathbf{C}^{*r} = (\mathbf{C}^{k+1}/\mathbf{C}^*)/\mathbf{C}^{*r-1}$; it is useful to introduce the $\mathcal{S}_{(a)} = \{x_i; q_i^a = 0; \quad 0 \leq i \leq k\}$ subspaces that are invariant under the a -th factor of the \mathbf{C}^{*r} group. So the manifold \mathcal{S} stable under \mathbf{C}^{*r} is given by the intersection of the various $\mathcal{S}_{(a)}$'s,

$$\mathcal{S} = \cap_{a=1}^r \mathcal{S}_{(a)} \tag{4.2}$$

If we suppose that $\{x_{i_0}; \dots; x_{i_{k_0-1}}\}$ those local coordinates that have non zero q_i^a charges and $\{x_{i_{k_0-1}}; \dots; x_{i_k}\}$ the coordinates that are fixed under \mathbf{C}^{*r} actions; then the manifold \mathcal{S} is given by,

$$\mathcal{S} = \{(0, \dots, 0, x_{k_0}, \dots, x_k)\} \subset \mathcal{V}_{d+1} \subset \mathbf{C}^{k+1} \tag{4.3}$$

To get the representation of the Z_i variable operators on the \mathcal{S} space, let us first consider what happens on its neighboring space $\mathcal{S}^\epsilon = \{(\epsilon, \dots, \epsilon, x_{k_0} + \epsilon, \dots, x_k + \epsilon)\}$, where we have taken $\epsilon_0 = \dots = \epsilon_k = \epsilon$ and where ϵ is as small as possible. Using the hypothesis $q_{i_{k_0}}^a = \dots = q_{i_k}^a = 0$ we have made and replacing the x_i moduli by their expression on \mathcal{S}^ϵ , then putting in the realization eqs(3.28), we get the following result,

$$Z_{i_j} = \epsilon \prod_{a=1}^r U_a^{q_{i_j}^a}; \quad 0 \leq j \leq k_0 - 1, \quad (4.4)$$

$$Z_{i_j} = (x_{i_j} + \epsilon) I_{id}; \quad k_0 \leq j \leq k. \quad (4.5)$$

The representation of the Z_i 's on the space \mathcal{S}_0 is then obtained by taking the limit $\epsilon \rightarrow 0$. As such non trivial operators are given by $Z_{i_j} \sim x_{i_j} I_{id}$; $k_0 \leq j \leq k$; they are proportional to the identity I_{id} operator of group representation $\mathcal{R}(C^{*r})$. This an important point since the Z_i operators are reducible into an infinite component sum as shown here below,⁴

$$Z_i = \sum_{\mathbf{n}} Z_i^{(\mathbf{n})}; \quad Z_i^{(\mathbf{n})} = x_i \pi_{\mathbf{n}}. \quad (4.6)$$

The above decomposition of the Z_i 's on \mathcal{S} has a nice interpretation in the D brane language. Thinking of the x_i variables as the coordinates of a D p brane, ($p = 2d$), wrapping the Calabi-Yau manifold M_d , it follows that, due torsion of the C^{*r} toric group, the D p brane at the singular points fractionate in the same manner as in the analysis of Berenstein and Leigh for the case of orbifolds with discrete isometries. In addition to the results of [24], which apply as well for the present study, there is a novelty here due to the dimension of the completely reducible $\mathcal{R}(C^{*r})$ group representation. There are infinitely many values for the C^* characters and so an infinite number of fractional $D(p - 2k_0)$ branes wrapping \mathcal{S} .

4.2 Quiver Diagrams

Like for the case of Calabi-Yau orbifolds with discrete symmetries, one can here also describe the varieties of fractional D branes by generalizing the Berenstein and Leigh quiver diagrams for \mathcal{M}_d^{nc} at the fixed points of the \mathbf{C}^{*r} toric actions. One of the basic ingredients in getting these graphs is the identification of the projectors of the \mathbf{C}^{*r} toric action group and the step operators \mathbf{a}^\pm acting as shift operators on the basis states of the group representation space. Since these actions are given by a kind of complexification of $U(1)^r$ and as each \mathbf{C}^* group factor has completely reducible representations with four possible sectors, it is interesting to treat separately these different cases. The various sectors for each \mathbf{C}^* subsymmetry factor are as follows; (i) (*dis, dis*) discrete-discrete sector where the C^* characters $\chi_n(\psi)$ are given $\chi_n(\psi) = \exp i\psi n$; $n \in \mathbf{Z} + i\mathbf{Z}$, $\psi \in \mathbf{C}$; (ii) (*dis, con*) discrete-continuous and (*con, dis*) continuous-discrete sectors and finally (iii) (*con, con*) continuous-continuous sector with characters $\chi(p, \psi)$ given by $\chi(p, \psi) = \exp ip\psi$; $p \in \mathbf{C}$.

Recall first of all that, due to torsion of the C^{*r} toric symmetries, the algebraic structure of the D p branes wrapping the compact manifold \mathcal{M}_d change. Brane points $\{x_i\}$ of commutative type *IIA* geometry become, in presence of torsion, fibers based on $\{x_i\}$. These fibers are valued in the algebra of

⁴The sums involved in the decomposition of the identity are either discrete series, integrals or both of them depending on whether the group representation $\mathcal{R}(C^{*r})$ spectrum is discrete, continuous or with discrete and continuous sectors. Therefore we have either $I_{id} = \sum_{\mathbf{n}} \pi_{\mathbf{n}}$, $\mathbf{n} = (n_1, \dots, n_r)$; $I_{id} = \int d\sigma \pi(\sigma)$, $\sigma = (\sigma_1, \dots, \sigma_r)$; or again $\sum_{\mathbf{m}} \int d\zeta \pi_{\mathbf{m}}(\zeta)$; Here $\pi_{\mathbf{m}}$, $\mathbf{m} = (n_{i_1}, \dots, n_{i_{r_0}})$ and $\pi(\zeta)$, $\zeta = (\zeta_{i_{r_0+1}}, \dots, \zeta_{i_r})$; and are the C^{*r} representation projectors considered in section 3.

the group representation $\mathcal{R}(C^{*r})$ and may be given a simple graph description on fixed spaces. While points $x_i.1$ in the commutative type *IIA* geometry are essentially numbers, the Z_i coordinate operators can be thought of, in the case of a discrete spectrum of \mathbf{C}^* , as

$$x_i.1 \rightarrow Z_i = (Z_i)_{mn} U^m V^n, \quad (4.7)$$

where U and V , $UV = e^{-i\psi} VU$, are the generators of the C^* toric group

Extending the results of [24], one can draw graphs for fractional D branes. Due to the decomposition of I_{id} eqs(3.13) and (3.20), we associate to each D p brane coordinate a quiver diagram mainly given by the product of (discrete or continuous) S^1 circles. For the simplest case $r = 1$ and C^* discrete representations, the quiver diagram is built as follows:

(1) To each $\pi_n = |n\rangle\langle n|$ projector it is associated a vertex point on a discrete S^1 circle. As there is an infinite number of points that one should put on S^1 , all happens as if the quiver diagram is given by the $\mathbf{Z}+i\mathbf{Z}$ lattice plus a extra point at infinity.

(2) The $a_n^\pm = |n \pm 1\rangle\langle n|$ shift operators are associated with the oriented links joining adjacent vertices, (vertex $(n-1)$ to the vertex (n) for a_n^- and vertex (n) to the vertex $(n+1)$ for a_n^+), of quiver diagram. They act as automorphisms exchanging the \mathbf{C}^* characters.

Moreover, as non zero D p brane coordinates at the singularities are of the form $Z_i \sim \sum_n Z_i^{(n)}$, it follows that D p branes on \mathcal{S} sub-manifolds fractionate into an infinite set of fractional $D2s$ branes coordinated by $Z_i^{(n)}$. This is a remarkable feature which looks like the inverse process of tachyon condensation mechanism à la GMS [8] where for instance a D p_1 brane on a NC Moyal plane decomposes into an infinite set of D $(p_1 - 2)$ branes. For details see studies on branes and Noncommutative Solitons [8],[30]-[32] in particular $D25$ branes decaying into an infinite $D23$ ones. In the case of a continuous spectrum, the corresponding quiver diagram is given by cross products of circles.

5 Conclusion

Using the algebraic geometry approach of Berenstein and Leigh, we have studied the type *IIA* geometry of non commutative Calabi-Yau manifolds embedded in non commutative toric varieties \mathcal{V} . Actually this study completes partial results of works in the literature on NC Calabi-Yau manifolds and too particularly orbifolds of Calabi-Yau homogeneous hypersurfaces with discrete torsion. Our construction has also the particularity of going beyond the idea of Berenstein and Leigh by introducing non commutative toric actions $(\mathbf{C}^{*r})^{nc}$ involving NC complex torii generalizing the Connes et al ones used in the study of matrix model compactification. From field theoretic point of view, our way of doing may be thought of as a step for approaching non commutative extension of supersymmetric gauged linear sigma models and their Landau Ginzburg mirrors.

The results established in this paper concerns non commutative extension of the class of Calabi-Yau manifolds \mathcal{M}_d embedded in toric varieties \mathcal{V}_{d+1} with \mathbf{C}^{*r} toric actions endowed by asymmetries. The latters are completely specified by the toric data

$$\{q_i^a; \nu_i^A; p_\alpha^I; \nu_{\alpha A}^*; \quad 0 \leq i \leq k; \quad 1 \leq a \leq r; \quad 1 \leq A \leq d+1\}, \quad (5.1)$$

with Calabi-Yau condition $\sum_{i=0}^k q_i^a = 0$, the toric geometry relations $\sum_{i=0}^k q_i^a \nu_i^A = 0$ and $\sum_{\alpha=0}^{d+1+r^*} p_\alpha^I \nu_{\alpha A}^* = 0$; $I = 1, \dots, r^*$. These eqs define the toric polygons of the variety \mathcal{V}_{d+1} . Non commutative structure is carried either by quantum symmetries described by inner automorphisms of \mathbf{C}^{*r} or again by considering NC complex cycles within the toric group in the same manner as one does in the Connes et al approach of toroidal compactification of matrix model of M theory. In our present case non commutative structure

is indeed solved in terms of asymmetries of the \mathbf{C}^{*r} toric actions and the toric data of the underlying \mathcal{V}_{d+1} . This result extends partial ones on NC geometries using discrete torsion of isometries of orbifolds of Calabi-Yau homogeneous hypersurfaces which are recovered as particular cases. Among our main results, we quote the two following:

(1) A class of complex d dimension NC Calabi-Yau manifolds \mathcal{M}_d^{nc} is naturally described in the language of toric geometry. They are given by subalgebras of NC toric varieties $\mathcal{V}_{d+1}^{nc} \sim \mathbf{C}_\theta^{k+1}/\mathbf{C}_{m,\tau}^{*r}$ where the NC matrix parameter θ_{ij} is induced by asymmetries of the \mathbf{C}^{*r} toric group. In this picture discrete groups Γ may be also included by taking into account the discrete symmetries of the toric variety generally described by $\prod_\alpha u_\alpha^{p_\alpha^I} = 1$. For both kinds of NC toric varieties $\mathcal{V}_{d+1}^{nc} \sim \mathbf{C}_\theta^{k+1}/\mathbf{C}_{m,\tau}^{*r}$ and $\mathcal{V}_{d+1}^{nc} \sim \mathbf{C}_\Theta^{k+1}/\mathbf{C}_{m,\tau}^{*r} \times \Gamma$, where in addition to θ_{ij} , the Θ_{ij} parameters have extra contributions coming from discrete torsion of Γ , we have

$$\mathcal{M}_d^{nc} \subset \mathcal{V}_{d+1}^{nc}; \quad \mathcal{M}_d = \mathcal{Z}(\mathcal{M}_d^{nc}) \subset \mathcal{V}_{d+1} = \mathcal{Z}(\mathcal{V}_{d+1}^{nc}) \quad (5.2)$$

Results obtained in this way covers as special cases those derived by following the method used in [15],[17],[24],[18] and where no reference to the toric data are made; see also [34].

(2) As far Calabi-Yau manifolds embedded in toric varieties are concerned, we have shown that the θ_{ij} parameters of the non commutative structure have contributions involving the toric data of polygons and, in addition to the Calabi-Yau condition $\sum_{i=0}^k q_i^a = 0$, it uses as well the relations $\sum_{i=0}^k q_i^a \nu_i^A = 0$ for the solving of the constraint eqs. Note in passing that in the analysis of section 3, we have considered the special projections of eq $\sum_{i=0}^k q_i^a \nu_i^A = 0$ on v_j^A and v_{jA}^* namely; $\sum_{i=0}^k q_i^a \nu_i^A \cdot v_j^A = 0$ and $\sum_{i=0}^k q_i^a \nu_i^A \cdot v_{jA}^* = 0$. In general one may also use other projections by generic vectors \mathbf{u}_A of the \mathbf{Z}^{d+1} lattice. An other remarkable feature of the \mathbf{C}^{*r} toric group is that at its fixed points we have an infinite set of fractional D branes instead of a finite one as it the case of NC orbifolds by \mathbf{Z}_N^n groups as shown in [15, 17]. This special feature is similar to the tachyon condensation picture of string field theory [8],[33].

To do so, we have first studied the type IIA geometry of complex d dimension Calabi-Yau manifolds using toric geometry methods. Then we have given the constraint eqs defining their NC geometry extensions by using the Berenstein *et al* method and second by considering embedding in NC toric varieties. To work out the regular solutions of the constraint eqs, we have developed two realizations of the NC \mathbf{C}^{*r} toric group; one involving quantum symmetries generated by shift operators V_a on the states of the \mathbf{C}^{*r} group representation and the other using the torsions between the generators U_a of the toric group. Next we have given different classes of solutions depending the nature of torsions of the \mathbf{C}^{*r} toric symmetries. As singular points of the toric actions are completely characterized by the q_i^a charges, we have studied also the singular representations of the constraint eqs and analyzed fractional D branes at singularities. Since the representations of the abelian \mathbf{C}^{*r} group are completely reducible, we have fractional D branes at the singularities; but with the remarkable feature that now there is an infinite set of them. This property follows naturally from the fact that identity I_{id} of the representation is decomposable into infinite sums over the π_n and $\pi(\sigma)$ projectors namely $I_{id} = \sum_n \pi_n$ for discrete spectrums and $I_{id} = \int d\sigma \pi(\sigma)$ for continuous ones; see also footnote 4. Actually this is a special feature of the non commutative structure induced by torsions of continuous \mathbf{C}^{*r} groups; it is related to the condensation phenomenon à la GMS considered few years ago in [8] and subsequent works. As a perspective, it would be interesting to analyze the properties of non commutative type IIB geometry dual to type IIA NC geometry considered in this paper; then explore the features of mirror symmetry in the case NC Calabi-Yau manifolds. This study will be developed in the second part of this work [35].

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